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Bosonisation and Soldering of Dual Symmetries in Two and Three Dimensions

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Abstract

We develop a technique that solders the dual aspects of some symmetry following from the bosonisation of two distinct fermionic models, thereby leading to new results which cannot be otherwise obtained. Exploiting this technique, the two dimensional chiral determinants with opposite chirality are soldered to reproduce either the usual gauge invariant expression leading to the Schwinger model or, alternatively, the Thirring model. Likewise, two apparently independent three dimensional massive Thirring models with same coupling but opposite mass signatures, in the long wavelegth limit, combine by the process of bosonisation and soldering to yield an effective massive Maxwell theory. The current bosonisation formulas are given, both in the original independent formulation as well as the effective theory, and shown to yield consistent results for the correlation functions. Similar features also hold for quantum electrodynamics in three dimensions.

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1 Introduction

Bosonisation is a powerful technique that maps a fermionic theory into its bosonic counterpart. It was initially developed and fully explored in the context of two dimensions[1]. More recently, it has been extended to higher dimensions[2, 3, 4, 5]. The importance of bosonisation lies in the fact that it includes quantum effects already at the classical level. Consequently, different aspects and manifestations of quantum phenomena may be investigated directly, that would otherwise be highly nontrivial in the fermionic language. Examples of such applications are the computation of the current algebra[4] and the study of screening or confinement in gauge theories[6].

This paper is devoted to analyse certain features and applications of bosonisation which, as far as we are aware, are unexplored even in two dimensions. The question we pose is the following: given two independent fermionic models which can be bosonised separately, under what circumstances is it possible to represent them by one single effective theory? The answer lies in the symmetries of the problem. Two distinct models displaying dual aspects of some symmetry can be combined by the simultaneous implementation of bosonisation and soldering to yield a completely new theory. This is irrespective of dimensional considerations. The technique of soldering essentially comprises in lifting the gauging of a global symmetry to its local version and exploits certain concepts introduced in a different context by Stone[7] and one of us[8]. The analysis is intrinsically quantal without having any classical analogue. This is easily explained by the observation that a simple addition of two independent classical lagrangeans is a trivial operation without leading to anything meaningful or significant.

The basic notions and ideas are first introduced in the context of two dimensions where bosonisation is known to yield exact results. The starting point is to take two distinct chiral lagrangeans with opposite chirality. Using their bosonised expressions, the soldering mechanism fuses, in a precise way, the left and right chiralities. This leads to a general lagrangean in which the chiral symmetry no longer exists, but it contains two extra parameters manifesting the bosonisation ambiguities. It is shown that different parametrisations lead to different models. In particular, the gauge invariant Schwinger model and Thirring model are reproduced. As a byproduct, the importance of Bose symmetry is realised and some interesting consequences regarding the arbitrary parametrisation in the chiral Schwinger model are charted.

Whereas the two dimensional analysis lays the foundations, the subsequent three dimensional discussion illuminates the full power and utility of the present approach.

While the bosonisation in these dimensions is not exact, nevertheless, for massive fermionic models in the large mass or, equivalently, the long wavelength limit, well defined local expressions are known to exist[3, 4]. Interestingly, these expressions exhibit a self or an anti self dual symmetry that is dictated by the signature of the fermion mass. Clearly, therefore, this symmetry simulates the dual aspects of the left and right chiral symmetry in the two dimensional example, thereby providing a novel testing ground for our ideas. Indeed, two distinct massive Thirring models with opposite mass signatures, are soldered to yield a massive Maxwell theory. This result is vindicated by a direct comparison of the current correlation functions obtained before and after the soldering process. As another instructive application, the fusion of two models describing quantum electrodynamics in three dimensions is considered. Results similar to the corresponding analysis for the massive Thirring models are obtained.

We conclude by discussing future prospects and possibilities of extending this analysis in different directions.

2   The two dimensional example

In this section we develop the ideas in the context of two dimensions. Consider, in particular, the following lagrangeans with opposite chiralities,

$$
\begin{aligned}
\mathcal{L}_+ &= \bar{\psi}(i\slashed{\partial} + e\slashed{A}P_+)\psi \\
\mathcal{L}_- &= \bar{\psi}(i\slashed{\partial} + e\slashed{A}P_-)\psi
\end{aligned}
\tag{1}
$$

where $P_\pm$ are the projection operators,

$$
P_\pm = \frac{1 \pm \gamma_5}{2}
\tag{2}
$$

It is well known that the computation of the fermion determinant, which effectively yields the bosonised expressions, is plagued by regularisation ambiguities since chiral gauge symmetry cannot be preserved[9]. Indeed an explicit one loop calculation following Schwinger's point splitting method [10] yields,

$$
\begin{aligned}
W_+[\varphi] &= -i\log\det(i\slashed{\partial} + e\slashed{A}_+) = \frac{1}{4\pi}\int d^2x \left(\partial_+\varphi\partial_-\varphi + 2\,e\,A_+\partial_-\varphi + a\,e^2\,A_+A_-\right) \\
W_-[\rho] &= -i\log\det(i\slashed{\partial} + e\slashed{A}_-) = \frac{1}{4\pi}\int d^2x \left(\partial_+\rho\partial_-\rho + 2\,e\,A_-\partial_+\rho + b\,e^2\,A_+A_-\right)
\end{aligned}
\tag{3}
$$

3

where the light cone metric has been invoked for convenience,

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_1) = A^\mp \quad ; \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1) = \partial^\mp \tag{4}$$

Note that the regularisation or bosonisation ambiguity is manifested through the arbitrary parameters a and b. The latter ambiguity is particularly transparent since by using the normal bosonisation dictionary $\bar{\psi} i \partial\!\!\!/ \psi \to \partial_+ \varphi \partial_- \varphi$ and $\bar{\psi} \gamma_\mu \psi \to \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi$ (which holds only for a gauge invariant theory), the above expressions with $a = b = 0$ are easily reproduced from (1).

It is crucial to observe that different scalar fields ϕ and ρ have been used in the bosonised forms to emphasize the fact that the fermionic fields occurring in the chiral components are uncorrelated. It is the soldering process which will abstract a meaningful combination of these components[11]. This process essentially consists in lifting the gauging of a global symmetry to its local version. Consider, therefore, the gauging of the following global symmetry,

$$\begin{aligned} \delta\varphi &= \delta\rho = \alpha \\ \delta A_\pm &= 0 \end{aligned} \tag{5}$$

The variations in the effective actions (3) are found to be,

$$\begin{aligned} \delta W_+[\varphi] &= \int d^2x \, \partial_- \alpha \, J_+(\varphi) \\ \delta W_-[\rho] &= \int d^2x \, \partial_+ \alpha \, J_-(\rho) \end{aligned} \tag{6}$$

where the currents are defined as,

$$J_\pm(\eta) = \frac{1}{2\pi}(\partial_\pm \eta + e \, A_\pm) \quad ; \quad \eta = \varphi, \rho \tag{7}$$

The important step now is to introduce the soldering field $B_\pm$ coupled with the currents so that,

$$W_\pm^{(1)}[\eta] = W_\pm[\eta] - \int d^2x \, B_\mp \, J_\pm(\eta) \tag{8}$$

Then it is possible to define a modified action,

$$W[\varphi, \rho] = W_+^{(1)}[\varphi] + W_-^{(1)}[\rho] + \frac{1}{2\pi} \int d^2x \, B_+ \, B_- \tag{9}$$

4

which is invariant under an extended set of transformations that includes (5) together with,

$$\delta B_\pm = \partial_\pm \alpha \tag{10}$$

Observe that the soldering field transforms exactly as a potential. It has served its purpose of fusing the two chiral components. Since it is an auxiliary field, it can be eliminated from the invariant action (9) by using the equations of motion. This will naturally solder the otherwise independent chiral components and justifies its name as a soldering field. The relevant solution is found to be,

$$B_\pm = 2\pi J_\pm \tag{11}$$

Inserting this solution in (9), we obtain,

$$W[\Phi] = \frac{1}{4\pi} \int d^2x \left\{ \left(\partial_+ \Phi \partial_- \Phi + 2\,e\,A_+ \partial_- \Phi - 2\,e\,A_- \partial_+ \Phi \right) + (a+b-2)\,e^2\,A_+\,A_- \right\} \tag{12}$$

where,

$$\Phi = \varphi - \rho \tag{13}$$

As announced, the action is no longer expreessed in terms of the different scalars φ and ρ, but only on their specific combination. This combination is gauge invariant.

Let us digress on the significance of the findings. At the classical fermionic version, the chiral lagrangeans are completely independent. Bosonising them includes quantum effects, but still there is no correlation. The soldering mechanism exploits the symmetries of the independent actions to precisely combine them to yield a single action. Note that the soldering works with the bosonised expressions. Thus the soldered action obtained in this fashion corresponds to the quantum theory.

We now show that different choices for the parameters a and b lead to well known models. To do this consider the variation of (12) under the conventional gauge transformations, $\delta\varphi = \delta\rho = \alpha$ and $\delta A_\pm = \partial_\pm \alpha$. It is easy to see that the expression in parenthesis is gauge invariant. Consequently a gauge invariant structure for W is obtained provided,

$$a + b - 2 = 0 \tag{14}$$

The effect of soldering, therefore, has been to induce a lift of the initial global symmetry (5) to its local form. By functionally integrating out the Φ field from (12),

we obtain,

$$W[A_+, A_-] = -\frac{e^2}{4\pi} \int d^2x \, \{A_+ \frac{\partial_-}{\partial_+} A_+ + A_- \frac{\partial_+}{\partial_-} A_- - 2A_+ A_-\} \qquad (15)$$

which is the familiar structure for the gauge invariant action expressed in terms of the potentials. The opposite chiralities of the two independent fermionic theories have been soldered to yield a gauge invariant action.

Some interesting observations are possible concerning the regularisation ambiguity manifested by the parameters a and b. Since a single equation (14) cannot fix both the parameters, it might appear that there is a whole one parameter class of solutions for the chiral actions that combine to yield the vector gauge invariant action. Indeed, without any further input, this is the only conclusion. However, Bose symmetry imposes a crucial restriction. This symmetry plays an essential part that complements gauge invariance. Recall, for instance, the calculation of the triangle graph leading to the Adler-Bell-Jackiw anomaly. The familiar form of the anomaly cannot be obtained by simply demanding gauge invariance; Bose symmetry at the vertices of the triangle must also be imposed[12, 13]. Similarly, Bose symmetry[14] is essential in reproducing the structure of the one-cocycle that is mandatory in the analysis on smooth bosonisation[15]; gauge invariance alone fails. In the present case, this symmetry corresponds to the left-right (or + -) symmetry in (3), thereby requiring $a = b$. Together with the condition (14) this implies $a = b = 1$. This parametrisation has important consequences if a Maxwell term was included from the beginning to impart dynamics. Then the soldering takes place among two chiral Schwinger models[16] having opposite chiralities to reproduce the usual Schwinger model[17]. It is known that the chiral models satisfy unitarity provided $a, b \geq 1$ and the spectrum consists of a vector boson with mass,

$$m^2 = \frac{e^2 a^2}{a - 1} \qquad (16)$$

and a massless chiral boson. The values of the parameters obtained here just saturate the bound. In other words, the chiral Schwinger model may have any $a \geq 1$, but if two such models with opposite chiralities are soldered to yield the vector Schwinger model, then the minimal bound is the unique choice. Moreover, for the minimal parametrisation, the mass of the vector boson becomes infinite so that it goes out of the spectrum. Thus the soldering mechanism shows how the massless modes in the chiral Schwinger models are fused to generate the massive mode of the Schwinger model.

Naively it may appear that the soldering of the left and right chiralities to obtain a gauge invariant result is a simple issue since adding the classical lagrangeans $\bar{\psi}\slashed{D}_+\psi$ and $\bar{\psi}\slashed{D}_-\psi$, with identical fermion species, just yields the usual vector lagrangean $\bar{\psi}\slashed{D}\psi$. The quantum considerations are, however, much involved. The chiral determinants, as they occur, cannot be even defined since the kernels map from one chirality to the other so that there is no well defined eigenvalue problem[18, 14]. This is circumvented by working with $\bar{\psi}(i\slashed{\partial} + e\slashed{A}_\pm)\psi$, that satisfy an eigenvalue equation, from which their determinants may be computed. But now a simple addition of the classical lagrangeans does not reproduce the expected gauge invariant form. At this juncture, the soldering process becomes important. It systematically combined the quantised (bosonised) expressions for the opposite chiral components. Note that *different* fermionic species were considered so that this soldering does not have any classical analogue, and is strictly a quantum phenomenon. This will become more transparent when the three dimensional case is discussed.

It is interesting to show that a different choice for the parameters a and b in (12) leads to the Thirring model. Indeed it is precisely when the mass term exists (*i.e.*, $a + b - 2 \neq 0$), that (12) represents the Thirring model. Consequently, this parametrisation complements that used previously to obtain the vector gauge invariant structure. It is now easy to see that the term in parentheses in (12) corresponds to $\bar{\psi}(i\slashed{\partial} + e\slashed{A})\psi$ so that integrating out the auxiliary A_μ field yields,

$$\mathcal{L} = \bar{\psi}i\slashed{\partial}\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2 \ \ ; \ g = \frac{4\pi}{a + b - 2} \tag{17}$$

which is just the lagrangean for the usual Thirring model. It is known [19]that this model is meaningful provided the coupling parameter satisfies the condition $g > -\pi$, so that,

$$\mid a + b \mid > 2 \tag{18}$$

This condition is the analogue of (14) found earlier. As usual, there is a one parameter arbitrariness. Imposing Bose symmetry implies that both a and b are equal and lie in the range

$$1 <\mid a \mid =\mid b \mid \tag{19}$$

This may be compared with the previous case where $a = b = 1$ was necessary for getting the gauge invariant structure. Interestingly, the positive range for the parameters in (19) just commences from this value.

Having developed and exploited the concepts of soldering in two dimensions, it is natural to investigate their consequences in three dimensions. The discerning reader

may have noticed that it is essential to have dual aspects of a symmetry that can be soldered to yield new information. In the two dimensional case, this was the left and right chirality. Interestingly, in three dimensions also, we have a similar phenomenon.

3 The three dimensional example

This section is devoted to an analysis of the soldering process in the massive Thirring model in three dimensions. We shall show that two apparently independent massive Thirring models in the long wavelength limit combine, at the quantum level, into a massive Maxwell theory. This is further vindicated by a direct comparison of the current correlation functions following from the bosonization identities. These findings are also extended to include three dimensional quantum electrodynamics. The new results and interpretations illuminate a close parallel with the two dimensional discussion.

3.1 The massive Thirring model

In order to effect the soldering, the first step is to consider the bosonisation of the massive Thirring model in three dimensions[3, 4]. This is therefore reviewed briefly. The relevant current correlator generating functional, in the Minkowski metric, is given by,

$$Z[\kappa] = \int D\psi D\bar{\psi} \exp\left(i \int d^3x \left[\bar{\psi}(i\not{\partial} + m)\psi - \frac{\lambda^2}{2} j_\mu j^\mu + \lambda j_\mu \kappa^\mu \right] \right) \qquad (20)$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$ is the fermionic current. As usual, the four fermion interaction can be eliminated by introducing an auxiliary field,

$$Z[\kappa] = \int D\psi D\bar{\psi} Df_\mu \exp\left(i \int d^3x \left[\bar{\psi} \left(i\not{\partial} + m + \lambda(\not{f} + \not{\kappa}) \right) \psi + \frac{1}{2} f_\mu f^\mu \right] \right) \qquad (21)$$

Contrary to the two dimensional models, the fermion integration cannot be done exactly. Under certain limiting conditions, however, this integation is possible leading

8

to closed expressions. A particularly effective choice is the large mass limit in which case the fermion determinant yields a local form. Incidentally, any other value of the mass leads to a nonlocal structure [5]. The large mass limit is therefore very special. The leading term in this limit was calculated by various means [20] and shown to yield the Chern-Simons three form. Thus the generating functional for the massive Thirring model in the large mass limit is given by,

$$Z[\kappa] = \int Df_\mu \exp\left(i \int d^3x \left(\frac{\lambda^2}{8\pi} \frac{m}{|m|} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda + \frac{1}{2} f_\mu f^\mu + \frac{\lambda^2}{4\pi} \frac{m}{|m|} \epsilon_{\mu\nu\sigma} \kappa^\mu \partial^\nu f^\sigma\right)\right)$$
(22)

where the signature of the topological terms is dictated by the corresponding signature of the fermionic mass term. In obtaining the above result a local counter term has been ignored. Such terms manifest the ambiguity in defining the time ordered product to compute the correlation functions[21]. The lagrangean in the above partition function defines a self dual model introduced earlier [22]. The massive Thirring model, in the relevant limit, therefore bosonises to a self dual model. It is useful to clarify the meaning of this self duality. The equation of motion in the absence of sources is given by,

$$f_\mu = -\frac{\lambda^2}{4\pi} \frac{m}{|m|} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda$$
(23)

from which the following relations may be easily verified,

$$\partial_\mu f^\mu = 0$$
$$\left(\Box + M^2\right) f_\mu = 0 \quad ; \quad M = \frac{4\pi}{\lambda^2}$$
(24)

A field dual to f_μ is defined as,

$$\tilde{f}_\mu = \frac{1}{M} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda$$
(25)

where the mass parameter M is inserted for dimensional reasons. Repeating the dual operation, we find,

$$\left(\tilde{\tilde{f}}_\mu\right) = \frac{1}{M} \epsilon_{\mu\nu\lambda} \partial^\nu \tilde{f}^\lambda = f_\mu$$
(26)

obtained by exploiting (24), thereby validating the definition of the dual field. Combining these results with (23), we conclude that,

$$f_\mu = -\frac{m}{|m|} \tilde{f}_\mu$$
(27)

9

Hence, depending on the sign of the fermion mass term, the bosonic theory corresponds to a self-dual or an anti self-dual model. Likewise, the Thirring current bosonises to the topological current

$$j_\mu = \frac{\lambda}{4\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\rho} \partial^\nu f^\rho \qquad (28)$$

The close connection with the two dimensional analysis is now evident. There the starting point was to consider two distinct fermionic theories with opposite chiralities. In the present instance, the analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures,

$$\mathcal{L}_+ = \bar{\psi}\,(i\not\partial + m)\,\psi - \frac{\lambda^2}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2$$
$$\mathcal{L}_- = \bar{\xi}\,(i\not\partial - m')\,\xi - \frac{\lambda^2}{2}\left(\bar{\xi}\gamma_\mu\xi\right)^2 \qquad (29)$$

Note that the only the relative sign between the mass parameters is important, but their magnitudes are different. From now on it is also assumed that both m and m' are positive. Then the bosonised lagrangeans are, respectively,

$$\mathcal{L}_+ = \frac{1}{2M}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda + \frac{1}{2}f_\mu f^\mu$$
$$\mathcal{L}_- = -\frac{1}{2M}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda + \frac{1}{2}g_\mu g^\mu \qquad (30)$$

where f_μ and g_μ are the distinct bosonic vector fields. The current bosonization formulae in the two cases are given by

$$j_\mu^+ = \bar{\psi}\gamma_\mu\psi = \frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu f^\rho$$
$$j_\mu^- = \bar{\xi}\gamma_\mu\xi = -\frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu g^\rho \qquad (31)$$

The stage is now set for soldering. Taking a cue from the two dimensional analysis, let us consider the gauging of the following symmetry,

$$\delta f_\mu = \delta g_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho\alpha^\sigma \qquad (32)$$

Under such transformations, the bosonised lagrangeans change as,

$$\delta\mathcal{L}_\pm = J_\pm^{\rho\sigma}(h_\mu)\partial_\rho\alpha_\sigma \quad ; \quad h_\mu = f_\mu,\ g_\mu \qquad (33)$$

10

where the antisymmetric currents are defined by,

$$J_{\pm}^{\rho\sigma}(h_\mu) = \epsilon^{\mu\rho\sigma}h_\mu \pm \frac{1}{M}\epsilon^{\gamma\rho\sigma}\epsilon_{\mu\nu\gamma}\partial^\mu h^\nu \tag{34}$$

It is worthwhile to mention that any other variation of the fields (like $\delta f_\mu = \alpha_\mu$)is inappropriate because changes in the two terms of the lagrangeans cannot be combined to give a single structure like (34). We now introduce the soldering field coupled with the antisymmetric currents. In the two dimensional case this was a vector. Its natural extension now is the antisymmetric second rank Kalb-Ramond tensor field $B_{\rho\sigma}$, transforming in the usual way,

$$\delta B_{\rho\sigma} = \partial_\rho\alpha_\sigma - \partial_\sigma\alpha_\rho \tag{35}$$

Then it is easy to see that the modified lagrangeans,

$$\mathcal{L}_{\pm}^{(1)} = \mathcal{L}_{\pm} - \frac{1}{2}J_{\pm}^{\rho\sigma}(h_\mu)B_{\rho\sigma} \tag{36}$$

transform as,

$$\delta\mathcal{L}_{\pm}^{(1)} = -\frac{1}{2}\delta J_{\pm}^{\rho\sigma}B_{\rho\sigma} \tag{37}$$

The final modification consists in adding a term to ensure gauge invariance of the soldered lagrangean. This is achieved by,

$$\mathcal{L}_{\pm}^{(2)} = \mathcal{L}_{\pm}^{(1)} + \frac{1}{4}B^{\rho\sigma}B_{\rho\sigma} \tag{38}$$

A straightforward algebra shows that the following combination,

$$\begin{aligned}
\mathcal{L}_S &= \mathcal{L}_+^{(2)} + \mathcal{L}_-^{(2)} \\
&= \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{2}B^{\rho\sigma}\left(J_{\rho\sigma}^+(f) + J_{\rho\sigma}^-(g)\right) + \frac{1}{2}B^{\rho\sigma}B_{\rho\sigma}
\end{aligned} \tag{39}$$

is invariant under the gauge transformations (32) and (35). The gauging of the symmetry is therefore complete. To return to a description in terms of the original variables, the auxiliary soldering field is eliminated from (39) by using the equation of motion,

$$B_{\rho\sigma} = \frac{1}{2}\left(J_{\rho\sigma}^+(f) + J_{\rho\sigma}^-(g)\right) \tag{40}$$

11

Inserting this solution in (39), the final soldered lagrangean is expressed solely in terms of the currents involving the original fields,

$$\mathcal{L}_S = \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{8}\left(J_{\rho\sigma}^+(f) + J_{\rho\sigma}^-(g)\right)\left(J_+^{\rho\sigma}(f) + J_-^{\rho\sigma}(g)\right) \tag{41}$$

It is now crucial to note that, by using the explicit structures for the currents, the above lagrangean is no longer a function of f_μ and g_μ separately, but only on the combination,

$$A_\mu = \frac{1}{\sqrt{2}M}\left(g_\mu - f_\mu\right) \tag{42}$$

with,

$$\mathcal{L}_S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A_\mu A^\mu \tag{43}$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{44}$$

is the usual field tensor expressed in terms of the basic entity A_μ. Our goal has been achieved. The soldering mechanism has precisely fused the self and anti self dual symmetries to yield a massive Maxwell theory which, naturally, lacks this symmetry.

It is now instructive to understand this result by comparing the current correlation functions. The Thirring currents in the two models bosonise to the topological currents (31) in the dual formulation. From a knowledge of the field correlators in the latter case, it is therefore possible to obtain the Thirring current correlators. The field correlators are obtained from the inverse of the kernels occurring in (30),

$$\langle f_\mu(+k)\, f_\nu(-k)\rangle = \frac{M^2}{M^2 - k^2}\left(ig_{\mu\nu} + \frac{1}{M}\epsilon_{\mu\rho\nu}k^\rho - \frac{i}{M^2}k_\mu\, k_\nu\right)$$

$$\langle g_\mu(+k)\, g_\nu(-k)\rangle = \frac{M^2}{M^2 - k^2}\left(ig_{\mu\nu} - \frac{1}{M}\epsilon_{\mu\rho\nu}k^\rho - \frac{i}{M^2}k_\mu\, k_\nu\right) \tag{45}$$

where the expressions are given in the momentum space. Using these in (31), the current correlators are obtained,

$$\langle j_\mu^+(+k)j_\nu^+(-k)\rangle = \frac{M}{4\pi(M^2 - k^2)}\left(ik^2 g_{\mu\nu} - ik_\mu\, k_\nu + \frac{1}{M}\epsilon_{\mu\nu\rho}k^\rho\, k^2\right)$$

$$\langle j_\mu^-(+k)j_\nu^-(-k)\rangle = \frac{M}{4\pi(M^2 - k^2)}\left(ik^2 g_{\mu\nu} - ik_\mu\, k_\nu - \frac{1}{M}\epsilon_{\mu\nu\rho}k^\rho\, k^2\right) \tag{46}$$

It is now feasible to construct a total current,

$$j_\mu = j_\mu^+ + j_\mu^- = \frac{\lambda}{4\pi}\epsilon_{\mu\nu\rho}\partial^\nu\left(f^\rho - g^\rho\right) \tag{47}$$

Then the correlation function for this current, in the original self dual formulation, follows from (46) and noting that $\langle j_\mu^+ j_\nu^- \rangle = 0$, which is a consequence of the independence of f_μ and g_ν;

$$\langle j_\mu(+k)\, j_\nu(-k)\rangle = \langle j_\mu^+ j_\nu^+ \rangle + \langle j_\mu^- j_\nu^- \rangle = \frac{iM}{2\pi(M^2 - k^2)}\left(k^2\, g_{\mu\nu} - k_\mu\, k_\nu\right) \tag{48}$$

The above equation is easily reproduced from the effective theory. Using (42), it is observed that the bosonization of the composite current (47) is defined in terms of the massive vector field A_μ,

$$j_\mu = \bar{\psi}\gamma_\mu\psi + \bar{\xi}\gamma_\mu\xi = -\sqrt{\frac{M}{2\pi}}\epsilon_{\mu\nu\rho}\partial^\nu A^\rho \tag{49}$$

The current correlator is now obtained from the field correlator $\langle A_\mu A_\nu \rangle$ given by the inverse of the kernel appearing in (43),

$$\langle A_\mu(+k)\, A_\nu(-k)\rangle = \frac{i}{M^2 - k^2}\left(g_{\mu\nu} - \frac{k_\mu\, k_\nu}{M^2}\right) \tag{50}$$

¿From (49) and (50) the two point function (48) is reproduced, including the normalization.

We conclude, therefore, that two massive Thirring models with opposite mass signatures, in the long wavelength limit, combine by the process of bosonisation and soldering, to a massive Maxwell theory. The bosonization of the composite current, obtained by adding the separate contributions from the two models, is given in terms of a topological current(49) of the massive vector theory. These are completely new results which cannot be obtained by a straightforward application of conventional bosonisation techniques. The massive modes in the original Thirring models are manifested in the two modes of (43) so that there is a proper matching in the degrees of freedom. Once again it is reminded that the fermion fields for the models are different so that the analysis has no classical analogue. Indeed if one considered the same fermion species, then a simple addition of the classical lagrangeans would lead to a Thirring model with a mass given by $m - m'$. In particular, this difference can be

13

zero. The bosonised version of such a massless model is known [2, 5] to yield a highly nonlocal theory which has no connection with (43). Classically, therefore, there is no possibility of even understanding, much less, reproducing the effective quantum result. In this sense the application in three dimensions is more dramatic than the corresponding case of two dimensions.

3.2   Quantum electrodynamics

An interesting theory in which the preceding ideas may be implemented is quantum electrodynmics, whose current correlator generating functional in an arbitrary covariant gauge is given by,

$$Z[\kappa] = \int D\bar{\psi} D\psi DA_\mu \, \exp \left\{ i \int d^3x \, \left(\bar{\psi} \left(i\slashed{\partial} + m + e\slashed{A} \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\eta}{2} (\partial_\mu A^\mu)^2 + e j_\mu \kappa^\mu \right) \right\} \tag{51}$$

where η is the gauge fixing parameter and $j_\mu = \bar{\psi} \gamma_\mu \psi$ is the current. As before, a one loop computation of the fermionic determinant in the large mass limit yields,

$$\begin{aligned} Z[\kappa] \;\; = \;\; & \int DA_\mu \, \exp \left\{ i \int d^3x \, \Big[\frac{e^2}{8\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \\ & + \;\; \frac{e^2}{4\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\rho} \kappa^\mu \partial^\nu A^\rho + \frac{\eta}{2} (\partial_\mu A^\mu)^2 \Big] \Big\} \end{aligned} \tag{52}$$

In the absence of sources, this just corresponds to the topolologically massive Maxwell-Chern-Simons theory, with the signature of the topological term determined from that of the fermion mass term. The equation of motion,

$$\partial^\nu F_{\nu\mu} + \frac{e^2}{4\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = 0 \tag{53}$$

expressed in terms of the dual tensor,

$$F_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{54}$$

reveals the self (or anti self) dual property,

$$F_\mu = \frac{4\pi}{e^2} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\lambda} \partial^\nu F^\lambda \tag{55}$$

14

which is the analogue of (23). In this fashion the Maxwell-Chern-Simons theory manifests the well known [23, 21, 24] mapping with the self dual models considered in the previous subsection. The difference is that the self duality in the former, in contrast to the latter, is contained in the dual field (54) rather than in the basic field defining the theory. This requires some modifications in the ensuing analysis. Furthermore, the bosonization of the fermionic current is now given by the topological current in the Maxwell-Chern-Simons theory,

$$j_\mu = \frac{e}{4\pi} \frac{m}{\mid m \mid} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{56}$$

Consider, therefore, two independent models describing quantum electrodynamics with opposite signatures in the mass terms,

$$
\begin{aligned}
\mathcal{L}_+ &= \bar{\psi}\left(i\not{\partial} + m + e\not{A}\right)\psi - \frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) \\
\mathcal{L}_- &= \bar{\xi}\left(i\not{\partial} - m' + e\not{B}\right)\xi - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B)
\end{aligned} \tag{57}
$$

whose bosonised versions in an appropriate limit are given by,

$$
\begin{aligned}
\mathcal{L}_+ &= -\frac{1}{4}F_{\mu\nu}(A) + \frac{M}{2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \quad ; \quad M = \frac{e^2}{4\pi} \\
\mathcal{L}_- &= -\frac{1}{4}F_{\mu\nu}(B) - \frac{M}{2}\epsilon_{\mu\nu\lambda}B^\mu\partial^\nu B^\lambda
\end{aligned} \tag{58}
$$

where A_μ and B_μ are the corresponding potentials. Likewise, the corresponding expressions for the bosonized currents are found from the general structure (56),

$$
\begin{aligned}
j_\mu^+ &= \bar{\psi}\gamma_\mu\psi = \frac{M}{e}\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda \\
j_\mu^- &= \bar{\xi}\gamma_\mu\xi = -\frac{M}{e}\epsilon_{\mu\nu\lambda}\partial^\nu B^\lambda
\end{aligned} \tag{59}
$$

To proceed with the soldering of the above models, take the symmetry transformation,

$$\delta A_\mu = \alpha_\mu \tag{60}$$

Such a transformation is spelled out by recalling (32) and the observation that now (54) simulates the f_μ field in the previous case. Under this variation, the lagrangeans (58) change as,

$$\delta\mathcal{L}_\pm = J_\pm^{\rho\sigma}(P)\partial_\rho\alpha_\sigma \quad ; \quad P = A, B \tag{61}$$

15

where the antisymmetric currents are defined by,

$$J^{\rho\sigma}_{\pm}(P) = \pm m\epsilon^{\rho\sigma\mu}P_\mu - F^{\rho\sigma}(P) \tag{62}$$

Proceeding as before, the antisymmetric soldering field $B_{\alpha\beta}$ transforming as (35) is introduced by coupling with these currents to define the first iterated lagrangeans analogous to (36),

$$\mathcal{L}^{(1)}_{\pm} = \mathcal{L}_{\pm} - \frac{1}{2}J^{\rho\sigma}_{\pm}(P)B_{\rho\sigma} \tag{63}$$

These lagrangeans are found to transform as,

$$\delta\mathcal{L}^{(1)}_{\pm} = \frac{1}{4}\delta B^2_{\lambda\sigma} - \frac{1}{2}\left(\pm m\epsilon_{\mu\lambda\sigma}\alpha^\mu B^{\lambda\sigma}\right) \tag{64}$$

It is now straightforward to deduce the final lagrangean that will be gauge invariant. This is given by,

$$\mathcal{L}_S = \mathcal{L}^{(2)}_+ + \mathcal{L}^{(2)}_- \quad ; \quad \delta\mathcal{L}_S = 0 \tag{65}$$

where the second iterated pieces are,

$$\mathcal{L}^{(2)}_{\pm} = \mathcal{L}_{\pm} - \frac{1}{2}J^{\rho\sigma}_{\pm}B_{\rho\sigma} - \frac{1}{4}B_{\rho\sigma}B^{\rho\sigma} \tag{66}$$

The invariance of $\mathcal{L}_S$ (65) is verified by observing that,

$$\delta\mathcal{L}^{(2)}_{\pm} = \mp\frac{1}{2}m\epsilon_{\mu\lambda\sigma}\alpha^\mu B^{\lambda\sigma} \tag{67}$$

To obtain the effective lagrangean it is necessary to eliminate the auxiliary $B_{\rho\sigma}$ field by using the equation of motion following from (65),

$$B_{\sigma\lambda} = -\frac{1}{2}\left(J^+_{\sigma\lambda}(A) + J^-_{\sigma\lambda}(B)\right) \tag{68}$$

Putting this back in (65), we obtain the final soldered lagrangean,

$$\mathcal{L}_S = -\frac{1}{4}F_{\mu\nu}(G)F^{\mu\nu}(G) + \frac{M^2}{2}G_\mu G^\mu \tag{69}$$

written in terms of a single field,

$$G_\mu = \frac{1}{\sqrt{2}}\left(A_\mu - B_\mu\right) \tag{70}$$

16

The lagrangean (69) governs the dynamics of a massive Maxwell theory.

As before, we now discuss the implications for the current correlation functions. These functions in the original models describing electrodynamics can be obtained from the mapping (59). The first step is to abstract the basic field correlators found by inverting the kernels occurring in (58). The results, in the momentum space, are

$$
\langle A_\mu(+k)\, A_\nu(-k) \rangle \;=\; \frac{i}{M^2 - k^2}\left[g_{\mu\nu} + \frac{M^2 - k^2(\eta+1)}{\eta k^4} k_\mu\, k_\nu + \frac{iM}{k^2}\epsilon_{\mu\rho\nu}k^\rho \right]
$$

$$
\langle B_\mu(+k)\, B_\nu(-k) \rangle \;=\; \frac{i}{M^2 - k^2}\left[g_{\mu\nu} + \frac{M^2 - k^2(\eta+1)}{\eta k^4} k_\mu\, k_\nu - \frac{iM}{k^2}\epsilon_{\mu\rho\nu}k^\rho \right] \quad (71)
$$

The current correlators are easily computed by substituting (71) into (59),

$$
\langle j_\mu^+(+k)\, j_\nu^+(-k) \rangle \;=\; i\left(\frac{M}{e}\right)^2 \frac{1}{M^2 - k^2}\left[k^2 g_{\mu\nu} - k_\mu\, k_\nu - iM\epsilon_{\mu\nu\rho}k^\rho \right]
$$

$$
\langle j_\mu^-(+k)\, j_\nu^-(-k) \rangle \;=\; i\left(\frac{M}{e}\right)^2 \frac{1}{M^2 - k^2}\left[k^2 g_{\mu\nu} - k_\mu\, k_\nu + iM\epsilon_{\mu\nu\rho}k^\rho \right] \quad (72)
$$

where, expectedly, the gauge dependent (η) contribution has dropped out. Defining a composite current,

$$
j_\mu = j_\mu^+ + j_\mu^- = \frac{M}{e}\epsilon_{\mu\nu\lambda}\partial^\nu\left(A^\lambda - B^\lambda\right) \quad (73)
$$

it is simple to obtain the relevant correlator by exploiting the results for j_μ^+ and j_μ^- from (72),

$$
\langle j_\mu(+k)\, j_\nu(-k) \rangle = 2i\left(\frac{M}{e}\right)^2 \frac{1}{M^2 - k^2}\left(k^2 g_{\mu\nu} - k_\mu\, k_\nu\right) \quad (74)
$$

In the bosonized version obtained from the soldering, (73) represents the mapping,

$$
j_\mu = \bar{\psi}\gamma_\mu\psi + \bar{\xi}\gamma_\mu\xi = \sqrt{2}\frac{M}{e}\epsilon_{\mu\nu\lambda}\partial^\nu G^\lambda \quad (75)
$$

where G_μ is the massive vector field (70) whose dynamics is governed by the lagrangean (69). In this effective description the result (74) is reproduced from (75) by using the correlator of G_μ obtained from (69), which is exactly identical to (50).

17

Thus the combined effects of bosonisation and soldering show that two independent quantum electrodynamical models with appropriate mass signatures are equivalently described by the massive Maxwell theory. In the self dual version the massive modes are the topological excitations in the Maxwell-Chern-Simons theories. These are combined into the two usual massive modes in the effective massive vector theory. A complete correspondence among the composite current correlation functions in the original models and in their dual bosonised description was also established. The comments made in the concluding part of the last subsection naturally apply also in this instance.

4   Conclusions

The present analysis clearly revealed the possibility of obtaining new results from quantum effects that conspire to combine two apparently independent theories into a single effective theory. The essential ingredient was that these theories must possess the dual aspects of the same symmetry. Then, by a systematic application of bosonisation and soldering, it was feasible to abstract a meaningful combination of such models, which can never be obtained by a naive addition of the classical lagrangeans.

The basic notions and ideas were particularly well illustrated in the two dimensional example where the bosonised expressions for distinct chiral lagrangeans were soldered to reproduce either the usual gauge invariant theory or the Thirring model. Indeed, the soldering mechanism that fused the opposite chiralities, clarified several aspects of the ambiguities occurring in bosonising chiral lagrangeans. It was clearly shown that unless Bose symmetry is imposed as an additional restriction, there is a whole one parameter class of bosonised solutions for the chiral lagrangeans that can be soldered to yield the vector gauge invariant result. The close connection between Bose symmetry and gauge invariance was thereby established, leading to a unique parametrisation. Similarly, using a different parametrisation, the soldering of the chiral lagrangeans led to the Thirring model. Once again there was a one parameter ambiguity unless Bose symmetry was imposed. If that was done, there was a specified range of solutions for the chiral lagrangeans that combined to yield a well defined Thirring model.

The elaboration of our methods was done by considering the massive version of the Thirring model and quantum electrodynamics in three dimensions. By the process of bosonisation such models, in the long wavelength limit, were cast in a

18

form which manifested a self dual symmetry. This was a basic perquisite for effecting the soldering. It was explicitly shown that two distinct massive Thirring models, with opposite mass signatures, combined to a massive Maxwell theory. The Thirring current correlation functions calculated either in the original self dual formulation or in the effective massive vector theory yielded identical results, showing the consistency of our approach. The application to quantum electrodynamics followed along similar lines.

It is evident that the present technique of combining models by the two step process of bosonisation and soldering can be carried through in higher dimensions provided the models have the relevant symmetry properties. It is also crucial to note that duality pervades the entire analysis. In the three dimensional case this was self evident since the models had a self (and anti) self dual symmetry. This was hidden in the two dimensional case where chiral symmetry was more transparent. But it may be mentioned that in two dimensions, chiral symmetry is the analogue of the duality $\partial_\mu \phi = \pm \epsilon_{\mu\nu} \partial^\nu \phi$. Interestingly, the duality in two dimensions was manifest in the lagrangeans while that in three dimensions was contained in the equations of motion. This opens up the possibility to discuss different aspects of duality, contained either in the lagrangean or in the equations of motion, in the same framework. Consequently, the methods developed here can be relevant and useful in diferent contexts; particularly in the recent discussions on electromagnetic duality or the study of chiral forms which exactly possess the type of self dual symmetry considered in this paper. We will report on these and related issues in a future work.

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